South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, Proceedings (2022), pp. 47-60

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ON HARMONIOUS LABELING FOR KRAGUJEVAC TREES AND REGULAR DENDRIMERS

Keerthi G. Mirajkar and Priyanka G. Sthavarmath

Department of Mathematics,
Karnatak University's Karnatak Arts College,
Dharwad - 580001, Karnataka, INDIA

E-mail : keerthi.mirajkar@gmail.com, priyankasthavarmath1990@gmail.com

(**Received:** *Apr. 08, 2022* **Accepted:** *Aug. 29, 2022* **Published:** *Aug. 30, 2022)*

Special Issue
Proceedings of National Conference on
"Emerging Trends in Discrete Mathematics, NCETDM - 2022"

Abstract: In this article, the results on harmonious and odd-harmonious labeling of Kragujevac trees and regular dendrimers are established.

Keywords and Phrases: Harmonious Labeling, Odd-harmonious Labeling, Kragujevac Trees ($Kg_{d,k}$) and Regular Dendrimers ($T_{k,d}$).

2020 Mathematics Subject Classification: 05C25, 05C05, 05C78.

1. Introduction and Preliminaries

One of the most fascinating field in graph theory is labeling of graph. In 1967, A. Rosa Studied graph labeling [8] and he conjectured that every tree is graceful. This concept has attracted a lot of attention in the last 50 years but has only been proved for some special classes of trees. Graph labeling is an allocation of numbers to $V(G)$ or $E(G)$ or both subject to certain conditions. In 1980, Graham and Slone [6] introduced the concept of harmonious labeling (HL) which is an vertex labeling function on additive bases. A labeling f on vertex set $(V(G))$ is considered to be harmonious, if f is an injective function from $V(G)$ to the group $(Z,+)$, hence the function f^* from $E(G)$ to $(Z,+)$ defined by $f^*(uv) = f(u)+f(v)(mod\ q)$ for each $uv \in E(G)$. If G is a tree then HL has same numbering

for two vertices [1]. Harmonious labeling of a graph plays an important role in social networking, rare probability events, spectral characterization of materials using X-ray crystallography and many more.
The odd-harmonious labeling [10] was introduced by Liang and Bai and defined as, G is considered to be odd-harmonious, if there exists an injection f from $V(G)$ to $\{0, 1, 2, \ldots, (2q-1)\}$ which induces f^* from $E(G)$ to $\{1, 3, 5, \ldots, (2q-1)\}$ given by $f^*(uv) = f(u) + f(v)(mod\ 2q)$ is bijective. If G admits odd-harmonious labeling then called odd-harmonious graph which are used for solution of undetermined equations. Many authors have investigated several different results for existence of both harmonious and odd-harmonious labelings [2, 9, 12]
Chemical graph theory [4] is a combination of chemistry and mathematics. An important part of this concept is molecular graph which represents the chemical compound in terms of a graph. The atom indicates the vertices and the chemical bond indicates the edges. Some structures such as silicate chains, benzenoid chains, regular dendrimer ($T_{k,d}$) etc., also form a type of molecular graph.
The noticed graphs here are simple, undirected, finite and connected. For undefined terminology and notations we refer [7] and for different labeling concepts we refer [3].

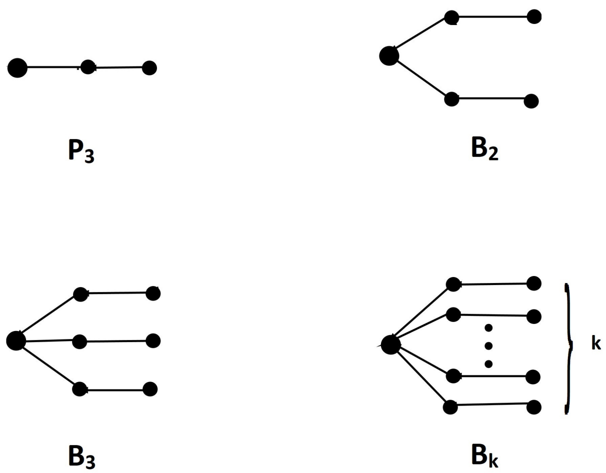

Figure 1: The rooted tree B_k

Let P_3 be a tree with three vertices rooted at one of its end vertex, the rooted tree B_k is the k copies of P_3. The rooted trees $B_2, B_3, B_4, ..., B_k$ are acquired respectively by noticing the roots of $2, 3$ and k copies of P_3. If $d \geq 2$ be an integer and $\beta_1, \beta_2, \beta_3, ..., \beta_d \in \{B_2, B_3, ..., B_k\}$ are rooted trees, then in **Kragujevac tree** ($Kg_{d,k}$), [4,5] T is a tree possessing a vertex of degree d which is central vertex adjacent to the roots of $\beta_1, \beta_2, \beta_3, ..., \beta_d$, where d is the degree of T and $\beta_1, \beta_2, \beta_3, ..., \beta_d$ are the branches of T. Here $\mid V(Kg_{d,k}) \mid = (2K + d + 1)$ and $\mid E(Kg_{d,k}) \mid = (2K + d)$ where K is an auxiliary parameter defined by $\sum k_i = K$, here k_i is total copies of P_3

The rooted trees B_2, B_3 and B_k are acquired respectively by identifying the roots of 2, 3 and k copies of P_3. Dendrimers are highly branched, mono-disperse macromolecules and it is a new class of polymeric materials and has a great impact on their physical and chemical properties which were first studied by Fritz Vogtle in 1978. A **regular dendrimer** $T_{k,d}$ [11, 13] for $k \geq 0$ and $d \geq 2$, is a tree with a center vertex u containing each non-pendant vertex of degree d and k is the distance from center vertex to each pendant vertex. Here $T_{k,d}$ has branches equal to its degree d and each branch of $T_{k,d}$ has $\dfrac{[(d-1)^k - 1]}{d-2}$ vertices.

2. Main Results

Results on harmonious labeling of Kragujevac trees and regular dendrimers are obtained.

Theorem 2.1. *For $2 \leq d \leq 3$ and $k \geq 2$ with isomorphic branches $\beta_{k_i}, i = 1, 2, 3, ...,$ then $Kg_{d,K}$, admits OHL.*

Proof. The Kragujevac tree $Kg_{d,K}$ with isomorphic branches β_{k_i} of degree d having $(2K + d + 1)$ vertices and $(2K + d)$ edges and every branch B_k having $(2k + 1)$ vertices and $2k$ edges. Let $\{u, u_1, v_1\} \cup \{u_i^1, u_i^2 \ /1 \leq i \leq n\} \cup \{v_i^1, v_i^2 \ /1 \leq i \leq m\} \cup \{w_i^1, w_i^2 \ /1 \leq i \leq l\}$ are the vertices of $Kg_{d,K}$, where n, m, and l are equal to $\dfrac{K}{2}$ and $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2(2K + d) - 1)\}$ the vertices are labeled as,

Case 1. The number of branches with $\dfrac{K}{2}$ copies of P_3 which is even with degree 2

$$\begin{aligned}
f(u) &= 0,\ f(u_1) = 1\ , f(v_1) = 3,\ f(u_i^1) = 2(i+1)\ \ for\ 1 \leq i \leq n \\
f(v_i^1) &= (f(u_n^1) + 2) + 4(i-1)\ \ for\ 1 \leq i \leq (m-1) \\
f(v_m^2) &= \mid E(G) \mid +7,\ f(u_1^2) = \mid V(G) \mid +2 \\
f(v_{m-i}^2) &= \mid V(Kg_{d,k}) \mid +2i\ \ for\ 1 \leq i \leq (m-1) \\
f(u_n^2) &= f^*(u_1 u_n^1),\ f(u_{n-1}^2) = f^*(u_1 u_{n-1}^1) \\
f(u_{n/2}^2) &= (2q-1) - f(u_{n/2}^1),\ f(u_i^2) = \mid V(Kg_{d,k}) \mid +2i\ for\ 1 \leq i \leq \frac{n}{2} - 1.
\end{aligned}$$

Case 2. The number of branches with $\frac{K}{2}$ copies of P_3 which is odd with degree 2

$$\begin{aligned}
f(u) &= 0,\ f(u_1) = 1\ ,f(v_1) = 3,\ f(u_i^1) = 2(i+1)\ \ for\ 1 \le i \le n \\
f(v_i^1) &= (f(u_n^1) + 4)) + 4i\ \ for\ 1 \le i \le m \\
f(v_m^2) &= f^*(u_1u_2^1),\ f(u_1^2) = \mid V(Kg_{d,k}) \mid +2,\ f(v_1^2) = (2q-1) - f(v_1^1) \\
f(v_{m-i}^2) &= (f^*(u_1u_1^1) + 2) + 2(i-1)\ for\ 1 \le i \le (\frac{m+1}{2}) - 1 \\
f(u_{m-(i-1)}^2) &= f^*(u_1u_n^1) - 2(i-1))\ for\ 1 \le i \le \frac{n+1}{2} \\
f(v_{\frac{m+1}{2}}^2) &= f^*(v_m^1v_m^2) - (m-1),\ f(u_i^2) = f(u_1^2) + 2i\ \ for\ \ 1 \le i \le (\frac{n+1}{2}) - 3
\end{aligned}$$

Case 3. The number of isomorphic branches of degree 3

$$\begin{aligned}
f(u) &= 0,\ f(u_1) = 1,\ f(v_1) = 3,\ f(w_1) = 5 \\
f(u_i^1) &= (f(w_1) + 1) + 4(i-1)\ \ for\ 1 \le i \le n \\
f(v_i^1) &= (f(u_n^1) + 2) + 4(i-1)\ \ for\ 1 \le i \le m \\
f(w_i^1) &= (f(v_m^1) + 2) + 2(i-1)\ \ for\ 1 \le i \le l \\
f(u_i^2) &= f^*(u_1u_1^1) + 4(i-1)\ \ for\ 1 \le i \le n \\
f(v_1^2) &= \mid E(Kg_{d,k}) \mid,\ f(v_i^2) = \mid E(Kg_{d,k}) \mid -2(i-1)\ for\ 2 \le i \le (m-1) \\
if\ d > 3\ \ f(v_m^2) &= 2 \mid E(Kg_{d,k}) \mid -(m+5) \\
if\ d = 3\ f(v_m^2) &= 2 \mid E(Kg_{d,k}) \mid -(K + (5n-9)) \\
if\ d = 3\ f(w_i^2) &= 2 \mid E(Kg_{d,k}) \mid -(8n-15) + 6(i-1),\ 1 \le i \le \frac{l+1}{2} \\
f(w_l^2) &= (2q-9),\ f(w_{l-2}^2) = 3l + 2 \\
if\ d > 3\ \ f(w_l^2) &= 2 \mid E(Kg_{d,k}) \mid -(8n-15),\ f(w_l^2) = 2 \mid E(Kg_{d,k}) \mid -(m-1)
\end{aligned}$$

The number of branches for both even and odd $\frac{K}{2}$ copies of P_3, the numbering of edges are as follows,

$$\begin{aligned}
f^*(u_1u) &= 1,\ f^*(v_1u) = 3,\ f^*(u_1u_i^1) = f^*(v_1u) + 2i,\ \ 1 \le i \le \frac{K}{2} \\
f^*(v_1v_i^1) &= (f(v^1) + 1_1) + 2i,\ \ 1 \le i \le \frac{K}{2} \\
f^*(u_i^1u_i^2) &= f(u_i^1) + f(u_i^2)\ \ (mod\ 2q),\ 1 \le i \le \frac{K}{2} \\
f^*(v_i^1v_i^2) &= f(v_i^1) + f(v_i^2)\ \ (mod\ 2q),\ 1 \le i \le \frac{K}{2}
\end{aligned}$$

The labels of edges of $Kg_{d,k}$ of degree 3 with isomorphic branches are as follows,

$$
\begin{aligned}
f^*(u_1u) &= 1,\ f^*(v_1u) = 3,\ f^*(w_1u) = 5 \\
f^*(u_1u_i^1) &= (f(u_1^1) + 1) + 4(i-1),\ ,\ 1 \le i \le \frac{K}{3} \\
f^*(v_1v_i^1) &= (f(v^1)_{\frac{K}{2}-(i-1)}) + 3,\ ,\ 1 \le i \le (\frac{K}{3} - 1) \\
f^*(w_1w_i^1) &= (f(w^1)_i) + 5,\ ,\ 1 \le i \le \frac{K}{3} \\
f^*(u_i^1u_i^2) &= f(u_i^1) + f(u_i^2)\ \ (mod\ 2q),\ 1 \le i \le \frac{K}{3} \\
f^*(v_i^1v_i^2) &= f(v_i^1) + f(v_i^2)\ \ (mod\ 2q),\ 1 \le i \le \frac{K}{3} \\
f^*(w_i^1w_i^2) &= f(w_i^1) + f(w_i^2)\ \ (mod\ 2q),\ 1 \le i \le \frac{K}{3}
\end{aligned}
$$

hence the arrangement of numbers of edges and vertices are not same. Thus $Kg_{d,K}$ admits odd hormonious labeling with $2 \le d \le 3$ of isomorphic branches.

Illustration 1. *OHL of $Kg_{d,K}$ which is with even and odd $\frac{K}{2}$ copies of P_3 is shown in fig 2. and OHL of $Kg_{d,K}$ with isomorphic branches of degree 3 is shown in fig 3.*

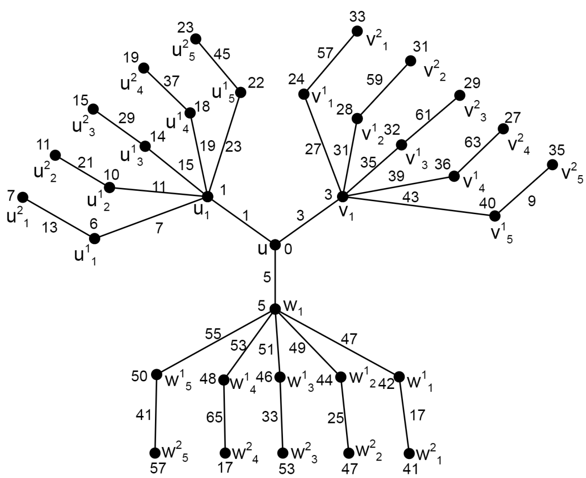

Figure 2: $Kg_{3,15}$

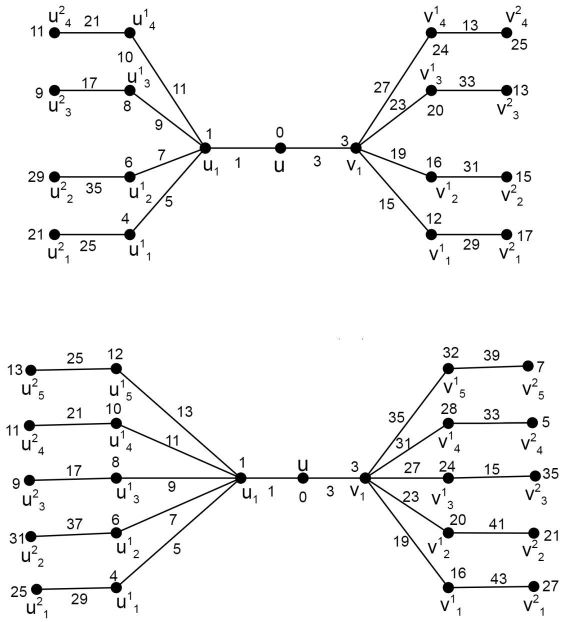

Figure 3: $Kg_{2,8}$ and $Kg_{2,10}$

Theorem 2.2. *For $d \geq 4$ with two stages and for $d = 3$ with four stages, the regular dendrimer $T_{k,d}$ admits OHL.*

Proof. Let $T_{k,d}$ be a tree with central vertex u of degree d and distance from central vertex is k. The total branches in $T_{k,d}$ are equal to its degree. B_k of $(T_{k,d})$ has $\dfrac{(d-1)^k - 1}{d-2}$ vertices and $\mid V(T_{k,d}) \mid = 1 + \dfrac{[d(d-1)^k - 1]}{d-2}$ and $\mid E(T_{k,d}) \mid = \dfrac{[d(d-1)^k - 1]}{d-2}$. Let $\{u\} \cup \{u_0^i \ /0 \leq i \leq (d-1)\} \cup \{u_1^i \ /1 \leq i \leq (d-1)\} \cup \{u_2^i, \ /1 \leq i \leq (d-1)\} \cup \{u_3^i \ /1 \leq i \leq (d-1)\}, ..., \{u_j^i, \ /0 \leq i, \ j \leq (d-1)\}$ are the vertices of $T_{k,d}$. Define $f : V(G) \to \{0, 1, 2, \ldots, (2(\mid E(T_{k,d}) \mid) - 1)\}$, where the labels of vertices are,

Case 1. For even degree

$$\begin{aligned}
f(u) &= 3, \ \ f(u_j) = 2j \ \ for \ \ 0 \leq j \leq (d-1) \\
f(u_1^1) &= f^*(uu_{d-1}), \ \ f(u_1^2) = f^*(u_{d-1}u_{d-1}^1) \\
f(u_{j+1}^1) &= f^*(uu_{d-1}) + 2j + 2(j-1) \ for \ 1 \leq j \leq (d-1) \\
f(u_{j+1}^2) &= f^*(u_{d-1}u_{d-1}^1) + 2j + 2(j-1) \ for \ 1 \leq j \leq (d-2) \\
f(u_{d-1}^3) &= f^*(uu_{d-2})
\end{aligned}$$

$$
\begin{aligned}
f(u_{d-2}^{i}) &= f^*(uu_{d-2}) + f(u_{d-1}) + 2(i-1) \;\; for \;\; d/2 \le i \le \frac{d}{2}+1 \\
if\ d &\ge 4 \\
f(u_{d-2}^{d/2}) &= f^*(uu_{d/2}),\; f(u_{d-1}^{d-1}) = f(u_{d-1}) + f^*(u_{d-2}u_{d-2}^{3}) \\
f(u_j^3) &= f^*(u_{d-1}u_{d-1}^{d-1}) + 2(i-1)\ for\ 1 \le j \le 3,\ 1 \le i \le 3.\ where\ i\ is\ odd \\
f(u_j^{\frac{d}{2}+1}) &= f^*(u_{d-3}u_{d-3}^{d/2}) + 2i + 2(i-1)\ for\ 1 \le i \le 2,\ 2 \le j \le 3 \\
f(u_2^{\frac{d}{2}+1}) &= f^*(u_{d-3}u_{d-3}^{d/2}),\;\; f(u_1^4) = f^*(u_{d-2}u_{d-2}^{d/2}) \\
f(u_j^{d-2}) &= f^*(u_{d-3}u_{d-3}^{4}) + 4(i+1)\ for\ 1 \le i \le 2,\ 2 \le j \le 4\ where\ j\ is\ even \\
f(u_j^{d-1}) &= f^*(u_{d-2}u_{d-2}^{\frac{d}{2}+1}) - 2(i+1) + 2i\ for\ 1 \le j \le 2,\ 1 \le i \le 3,\ where\ j\ is\ odd \\
f(u_2^{d-1}) &= f^*(u_{d-2}u_{d-2}^{\frac{d}{2}+1}),\; f(u_i^{d-1}) = f^*(u_{d-2}u_{d-2}^{\frac{d}{2}+2}) + 2i,\ 1 \le i \le 3 \\
f(u_{d-2}^{d-1}) &= f^*(u_{d-1}u_{d-1}^{d-2})
\end{aligned}
$$

Case 2. For odd degree

$$
\begin{aligned}
f(u) &= 3,\;\; f(u_j) = 2j \;\; for \;\; 0 \le i \le (d-1) \\
f(u_1^1) &= f^*(uu_{d-1}),\;\; f(u_{d-1}^4) = f(u_{d-2} + f^*(uu_{d-1}) \\
f(u_j^1) &= f^*(u_{j-1}u_{j-2}^2)\ for\ 2 \le j \le (d-1) \\
f(u_j^2) &= f(u_j^1) + f(u_j)\ for\ 1 \le j \le (d-1) \\
f(u_{d-1}^3) &= f^*(uu_{d-2}),\; f(u_{d-1}^4) = f^*(uu_{d-2}) + f(u_{d-1}) \\
f(u_1^3) &= f^*(u_{d-1}u_{d-1}^4) \\
f(u_j^3) &= f^*(u_{d-1}u_{d-1}^4) + 2j\ for\ 1 \le j \le (d-1),\ j\ is\ odd \\
f(u_j^3) &= f^*(u_{d-1}u_{d-1}^4) + 2j\ for\ 2 \le j \le (d-1),\ j\ is\ even \\
f(u_j^i) &= f^*(u_{d-2}u_{d-3}^3) + 2(i-4)\ for\ 2 \le i \le (d-1),\ 1 \le j \le 3,\ j\ is\ odd \\
if\ d &\ge 5, \\
f(u_{d-2}^3) &= f^*(uu_{d-3}),\; f(u_{d-2}^4) = f^*(uu_{d-3}) + f(u_{d-2}) \\
f(u_2^3) &= f^*(u_{d-2}u_{d-2}^4) \\
f(u_j^3) &= f^*(u_{d-2}u_{d-2}^4) + 2i + 2(i-1)\ for\ 1 \le i \le 2,\ 3 \le j \le (d-3) \\
f(u_{d-1}^{d-2}) &= f^*(u_{d-3}u_{d-3}^3) + f(u_{d-2}) \\
f(u_{d-2}^{d-2}) &= f^*(u_{d-3}u_{d-3}^3),\; f(u_1^3) = f^*(u_{d-1}u_{d-1}^3)
\end{aligned}
$$

$$\begin{aligned}
f(u_1^4) &= f^*(u_{d-3}u_{d-3}^4),\ f(u_3^{d-2}) = f^*(u_{d-1}u_{d-1}^{d-2}) \\
f(u_j^4) &= f^*(u_{d-1}u_{d-1}^3) + 2i + 2(i-1)\ for\ 1 \le i \le (d-3),\ 2 \le j \le (d-3) \\
f(u_j^4) &= f^*(u_{d-3}u_{d-3}^4) + 2(i-1)\ for\ \ 1 \le j \le 3\ j\ is\ odd \\
f(u_{d-1}^{d-2}) &= f^*(u_{d-1}u_{d-1}^{d-2}),\ f(u_{d-2}^{d-2}) = f^*(u_{d-3}u_{d-3}^{d-3}) \\
f(u_1^5) &= f^*(u_{d-2}u_{d-2}^{d-1}),\ f(u_2^{d-1}) = f^*(u_{d-2}u_{d-2}^{d-2}) + 2 \\
f(u_{d-1}^{d-1}) &= f^*(u_2u_2^{d-1}),\ f(u_3^{d-3}) = f^*(u_{d-2}u_{d-2}^6) \\
f(u_j^6) &= f^*(u_{d-1}u_{d-1}^{d-2}) + 2(i-1)\ for\ 1 \le i \le 2,\ 2 \le j \le 3,\ j\ is\ odd
\end{aligned}$$

Case 3. $T_{k,d}$ of degree 3 with four stages

$$\begin{aligned}
f(u) &= 3,\ f(u_j) = 2j\ \ for\ \ 0 \le j \le d \\
f(u_2^i) &= f^*(uu_3) + 2(l-1)\ for\ 1 \le l \le 2,\ 1 \le i \le 2 \\
f(u_2^i) &= 4f^*(u_2u_2^1) - 10(l-1)\ for\ 1 \le l \le 2,\ 3 \le i \le 5 \\
f(u_2^i) &= f(u_2^2) + l\ \ for\ 3 \le l \le 5,\ 4 \le i \le 6 \\
f(u_3^i) &= f^*(uu_3) + 4l\ for\ 1 \le l \le 2,\ 1 \le i \le 2 \\
f(u_3^i) &= f^*(u_3u_3^1) + l\ for\ 1 \le l \le 3,\ 3 \le i \le 5\ where\ l\ and\ i\ are\ odd \\
f(u_3^i) &= f(u_3^5 + 5) + (l-1)\ for\ 2 \le l \le 3,\ 4 \le i \le 6 \\
f(u_1^i) &= f^*(u_2u_2^1) + 4l\ for\ 1 \le l \le 2,\ 1 \le i \le 2 \\
f(u_1^i) &= f^*(u_1u_1^1) + l\ \ for\ \ 3 \le l \le 5,\ 3 \le i \le 5,\ where\ l\ and\ i\ are\ odd \\
f(u_1^i) &= 2l,\ for\ 4 \le l \le (d+2),\ 4 \le i \le (d+3) \\
f(u_1^i) &= (n+9) + 2(l-1)\ for\ 1 \le l \le (d+1),\ 8 \le i \le 14 \\
f(u_1^{13}) &= f^*(u_1^6u_1^{12}),\ f(u_1^9) = (n-1),\ f(u_3^{13}) = 39 \\
f(u_1^{11}) &= f^*(u_1^6u_1^{12}),\ f(u_1^7) = f^*(u_1^6u_1^{14}) \\
f(u_2^i) &= f^*(u_3u_3^1) + 2(l-1)\ for\ 1 \le l \le 2,\ 7 \le i \le 9, \\
f(u_2^i) &= (f^*(u_3u_3^1) + 6) + 2(l-1)\ for\ 1 \le l \le 2,\ 11 \le i \le 13 \\
f(u_1^{14}) &= n+1,\ f(u_2^i) = f(u_2^{13}) + 2l\ \ for\ 1 \le l \le d,\ 8 \le i \le 10 \\
f(u_2^{12}) &= (n+1) + 10,\ f(u_3^{11}) = f(u_1^{14}) - 2 \\
f(u_3^{12}) &= f^*(u_3^2u_3^6) + 2,\ f(u_3^{14}) = f^*(u_3^2u_3^6) \\
f(u_3^i) &= f(u_1^{14}) + 2 + 2(l-1)\ for\ 1 \le l \le 2,\ 7 \le i \le 9 \\
f(u_3^{10}) &= (n+3),\ f(u_3^8) = (n+3) + 2.
\end{aligned}$$

For both even and odd degree of $T_{k,d}$, the edges are labeled by

$$\begin{aligned}
f^*(uu_i) &= f(u) + 2(i-1),\ 0 \le i \le (d-1) \\
f^*(u_0u_0^j) &= f(u_0) + f(u_0^j)\ \ (mod\ 2q),\ 1 \le j \le (d-1)
\end{aligned}$$

$$\begin{aligned} f^*(u_1u_1^j) &= f(u_1) + f(u_1^j) \ (mod\ 2q),\ 1 \le j \le (d-1) \\ f^*(u_2u_2^j) &= f(u_2) + f(u_2^j) \ (mod\ 2q),\ 1 \le j \le (d-1) \\ &\vdots \\ f^*(u_iu_i^j) &= f(u_i) + f(u_i^j) \ (mod\ 2q),\ 0 \le i \le (d-1)\ and\ 1 \le j \le (d-1) \end{aligned}$$

The edge labels of case 3 are as follows, $f^*(uu_i) = f(u) + 2(i-1),\ 0 \le i \le d$, remaining edges of $T_{k,d}$ of degree 3 are labeled by taking sum of labels of terminal vertices of the edges which are distinctly labeled and f^* is bijective. So, the labeling of both vertices and edges are distinct. Hence the regular dendrimer $T_{k,d}$ of degree d admits odd hormonious labeling.

Illustration 2. *OHL of $T_{k,d}$ dendrimer with even and odd degree at the centre is shown in fig 4. and OHL of $T_{k,d}$ of degree 3 with 46-vertices in 4^{th} stage is illustrated in fig 5.*

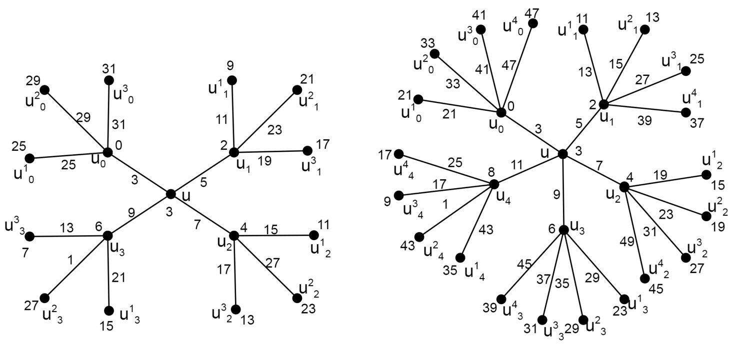

Figure 4: Regular dendrimers $T_{2,4}$ and $T_{2,5}$

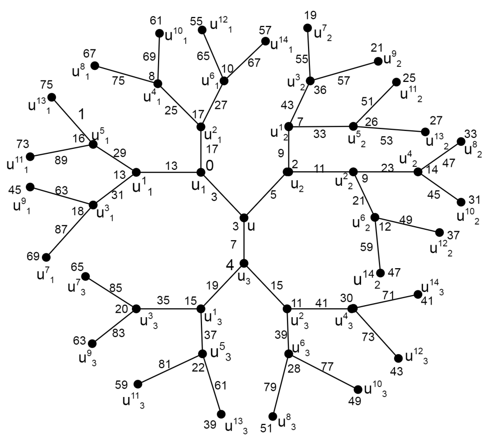

Figure 5: Regular dendrimer $T_{4,3}$

Theorem 2.3. *The Kragujevac tree of degree d with $2 \leq d \leq 5$ admits HL.*
Proof. The Kragujevac tree $Kg_{d,K}$ with isomorphic branches β_{k_i} of degree d with $(2K + d + 1)$ vertices and $(2K + d)$ edges. Let $\{u\} \cup \{u_1^i \ /1 \leq i \leq K\} \cup \{v_1^i, \ ;1 \leq i \leq K\}$ are the vertices of $Kg_{d,K}$ with $f : V(G) \to Z_q$, where the vertices are labeled as,

$$
\begin{aligned}
f(u) &= 0, \ f(u_i) = i \ \ for \ \ 1 \leq i \leq d \\
f(u_1^j) &= (f^*(uu_d) + 1) + (i-1) \ for \ 1 \leq i \leq 3, \ 1 \leq j \leq 3 \\
f(u_1^j) &= f(u_1^3) + 2i \ for \ 1 \leq i \leq 3, \ 4 \leq j \leq 5 \\
f(u_1^{d+2}) &= f^*(uu_1^{d+1}) + f(u_3) \\
f(u_1^{d+i}) &= f^*(u_3 u_1^{d+2}) + (i-1) \ for \ 1 \leq i \leq 2 \\
f(u_1^{2d-1}) &= 2d, \ \ f(u_1^{2d}) = f^*(u_1^{2d-1} u_d) \\
f(v_1^j) &= (f(v_1^1) - 2) + 3(i-1) \ for \ 1 \leq i \leq (d-1), \\
when \ d &= 4 \ and \ 2 \leq j \leq (d+2), \ for \ even \ j, f(v_1^3) = f(u_1^3) + 1 \\
f(v_1^j) &= f(v_1^1) + (\frac{d+1}{2} - 1) + 2(i-1), \\
when \ d &= 3 \ and \ 5 \ for \ 1 \leq i \leq 2, \ 4 \leq j \leq 6 \\
f(v_1^{2d-1}) &= 2d + 2, \ f(v_1^{2d}) = f^*(u_1^{2d} v_1^{2d-1}), \ when \ d = 4 \\
f(v_1^{2d-1}) &= (4d+1), \ f(v_1^{2d}) = f(v_1^{2d-1}) + 1 \ if \ d = 3 \ and \ 5 \\
f(u_1^{d+i}) &= f^*(u_1^{d+1} u_3) + (i-1) \ for \ 1 \leq i \leq 2 \ if \ d = 5 \\
f(v_1^{d+i}) &= f^*(v_1^{d+1} v_3) + (i-2) \ for \ 2 \leq i \leq 3 \ if \ d = 5.
\end{aligned}
$$

The labels of edges of $Kg_{d,k}$ are

$$
\begin{aligned}
f^*(uu_i) &= i, \ 1 \leq i \leq d, \ f^*(u_1 u_1^1) = 6 \\
f^*(u_1 u_1^i) &= (f^*(u_1 u_1^1) + 1) + 2(i-1), \ 1 \leq i \leq (d-1)
\end{aligned}
$$

The remaining edges are labeled by taking sum of labels of terminal vertices of edges which are distinctly labeled and f^* is bijective. Hence both the labeling pattern of edges and vertices are distinct. Thus $Kg_{d,K}$ admits hormonious labeling with $2 \leq d \leq 5$ and isomorphic branches .

Illustration 3. *Kragujevac tree of degree 4 is illustrated in the fig 6.*

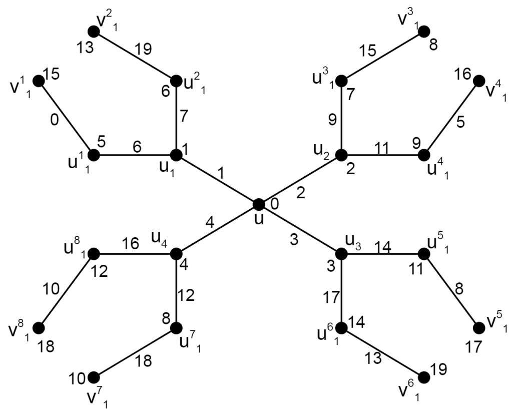

Figure 6: Kragujevac tree $Kg_{4,8}$

Theorem 2.4. *For $3 \leq d \leq 5$, the regular dendrimer $T_{k,d}$ of degree d and distance from central vertex k admits HL.*

Proof. Let $T_{k,d}$ be a tree with central vertex v of degree d with distance from central vertex k. The total branches in $T_{k,d}$ are equal to its degree. Each branch of $(T_{k,d})$ has $\dfrac{(d-1)^k-1}{d-2}$ vertices and $\mid V(T_{k,d}) \mid = 1 + \dfrac{[d(d-1)^k-1]}{d-2}$ and $\mid E(T_{k,d}) \mid = \dfrac{[d(d-1)^k-1]}{d-2}$. Let $\{u\} \cup \{u_0^i\ /0 \leq i \leq (d-1)\} \cup \{u_1^i,\ 1 \leq i \leq (d-1)\} \cup \{u_2^i,\ 1 \leq i \leq (d-1)\} \cup, ..., \{u_j^i,\ 0 \leq i, j \leq (d-1)\}$ are the vertices of $T_{k,d}$ and $f : V(G) \rightarrow Z_q$, such that the vertices are labeled as,

$$\begin{aligned}
f(u) &= d - \frac{d+1}{2},\ if\ d\ is\ odd\ and\ d - \frac{d+2}{2},\ if\ d\ is\ even \\
f(u_j) &= j\ for\ 0 \leq j \leq (d-1),\ f(u_1^1) = f^*(uu_{d-1}),\ f(u_1^2) = f(u_1^1) + 1 \\
f(u_j^1) &= f(u_1^1) + 2i + 2(i-1)\ for\ 1 \leq i \leq (d-2),\ and\ 2 \leq j \leq (d-1) \\
f(u_j^2) &= f(u_1^2) + 3i + 2(i-1)\ for\ 1 \leq i \leq (d-3),\ and\ 2 \leq j \leq (d-2) \\
f(u_{d-1}^2) &= f^*(u_{d-1}u_{d-1}^1),\ f(u_{d-1}^{d-2}) = f^*(uu_{d-2}) \\
f(u_{d-1}^{d-1}) &= f^*(uu_{d-1}^{d-2}),\ f(u_1^3) = f^*(u_{d-1}u_{d-1}^{d-1}) \\
f(u_j^3) &= f(u_1^3) + i\ for\ 1 \leq i \leq (\frac{d+1}{2} - 1),\ and\ 2 \leq j \leq \frac{d+1}{2} \\
f(u_1^{d-1}) &= f^*(u_{d-2}u_{d-3}^3),\ f(u_{d-2}^{d-1}) = f^*(u_1u_1^{d-1}),\ f(u_2^{d-1}) = f(u_3u_{d-2}^{d-1}) \\
if\ d = 4,\ f(u_1^3) &= 9,\ f(u_2^3) = 6,\ f(u_{d-1}^{d-2}) = f^*(u_1u_1^{d-1}),\ f(u_{d-1}^{d-1}) = 2.
\end{aligned}$$

$$f(u_j^1) = f(u_1^1) + 2i + 3(i-1) \; for \; 1 \leq i \leq 2, \; and \; 2 \leq j \leq 3$$
$$f(u_j^2) = f^*(u_{d-1}u_{d-1}^1) + (i-1) \; for \; 1 \leq i \leq 2, \; and \; 2 \leq j \leq 3$$

The edges labels of $T_{k,d}$ of degree d are,

$$f^*(uu_0) = 1, \; f^*(uu_i) = 3 + (i-1), \; 1 \leq i \leq (d-1)$$
$$f^*(u_0u_0^j) = f(u_0) + f(u_0^j) \; (mod \; 2q), \; 1 \leq j \leq (d-1)$$

$$f^*(u_1u_1^j) = f(u_1) + f(u_1^j) \; (mod \; 2q), \; 1 \leq j \leq (d-1)$$
$$f^*(u_2u_2^j) = f(u_2) + f(u_2^j) \; (mod \; 2q), \; 1 \leq j \leq (d-1)$$
$$\vdots$$
$$f^*(u_5u_5^j) = f(u_i) + f(u_5^j) \; (mod \; 2q), \; 1 \leq j \leq (d-1)$$

So, the labeling of both vertices and edges are distinct and the regular dendrimer $T_{k,d}$ of degree d with $3 \leq d \leq 5$ admits hormonious labeling.

Illustration 4. *HL of regular dendrimer $T_{k,d}$ of degree d with central vertex k is shown in fig 7.*

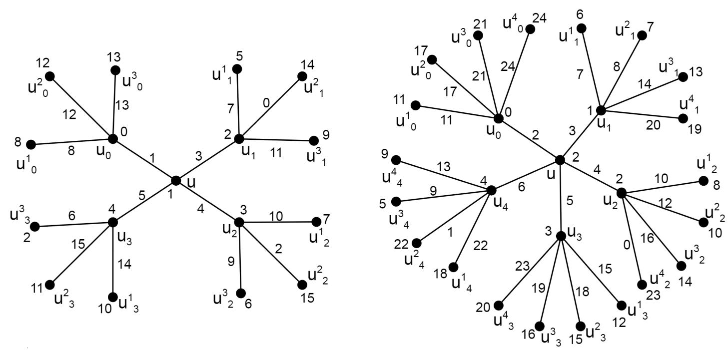

Figure 7: regular dendrimers $T_{2,4}$ and $T_{2,5}$

3. Conclusion

In this article, we have studied on vertex labeling functions called HL and OHL and obtained the results for Kragujevac trees and one of the molecular structure namely regular dendrimer which admits both HL and OHL.

Acknowledgement

Authors are thankful to Karnatak University, Dharwad, Karnataka, India for the support through University Research Studentship (URS), No. KU/Sch/URS/2021/623, dated: 26/10/2021.

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